The easiest way to prove that the obvious presentation of a semi-direct product is actually a presentation is to use von Dyck's lemma. Here is the argument and similar arguments can be used in most cases.

So suppose that G and H are presented as follows

$$G =< A; R >, \; H =< B; S >,$$

where

$$A = \{a_i \mid i \in I\}, \; B = \{b_j \mid j \in J.$$

Furthermore suppose ϕ is a homomorphism of H into the automorphism group of G. Let

$$E = G \rtimes_\phi H.$$

So identifying G and H with their images in E, we find that

$$b_j^{-1} a_i b_j = a_i(b_j \phi) = w_{i,j}(\underset{\rightarrow}{a}).$$

Now let $X = \{x_i \mid i \in I\}$ and let $Y = \{y_j \mid j \in J\}$ and let W be the group defined by the presentation

$$W =< X, Y; \{r(\underset{\rightarrow}{x}) \mid r \in R\} \cup \{s(\underset{\rightarrow}{y}) \mid s \in S\} \cup \{y_j^{-1} x_i y_j = w_{i,j}(\underset{\rightarrow}{x}) \mid j \in J, \, i \in I\} > .$$

Think of the elements of X and Y as elements of the group W. By von Dyck's lemma the mapping which sends x_i to a_i and y_j to b_j can be extended to a homomorphism ψ from W onto E. Now it follows from the defining relation of W that the elements $w \in W$ can be written as products which take the form $w = vu$ where $v = v(\underset{\rightarrow}{y})$ is a word in the generators coming from Y and $u = u(\underset{\rightarrow}{x})$ is a word in the generators coming from X. The claim now is that ψ is an isomorphism which will imply that W gives us a presentation for E. Suppose then that $w\psi = 1$. Notice that ψ maps vu onto $v(\underset{\rightarrow}{b})u(\underset{\rightarrow}{x})$ and since $w\phi = 1$, $v(\underset{\rightarrow}{b} = 1$ and $u(\underset{\rightarrow}{a} = 1$. Since the presentation for W contains the defining relations for both H and G it follows that $v(\underset{\rightarrow}{y}) = 1$ and $u(\underset{\rightarrow}{x}) = 1$ and so $w = 1$ as desired.