

The calculus of presentations and von Dyck's lemma: Lecture 4

Gilbert Baumslag

© Draft date October 3, 2008

October 3, 2008

1 Recalling the notion of a presentation and some definitions

Given a group G , we write

$$G = \langle X; R \rangle$$

where X is a set and R is a subset of the free group $F = F(X)$ on X , if G comes equipped with a map ϕ , often implicit, called a *presentation map*, from X to G such that the extension ϕ_* of ϕ to F is onto G with kernel $K = gp_F(R)$. So

$$G \cong F/K.$$

Now given a presentation $\langle X; R \rangle$, i.e., a pair consisting of a set X and a subset R of $F = F(X)$, there exists a group G with

$$G = \langle X; R \rangle .$$

This is accomplished as follows. First we define $| X; R |$, which we term *the group of $\langle X; R \rangle$* by

$$| X; R | = F/K,$$

where $F = F(X)$ and $K = gp_F(R)$. In order for this to be a presentation we need the presentation map ϕ . To this end, set

$$| x | = xK.$$

Then $\phi : x \mapsto | x |$ is a map from X to $| X; R |$ such that its extension to F has kernel $gp_F(R)$. We allow $R = \emptyset$ in which case G is simply the free group on X .

Each element $r \in R$ is termed a *relator* in G and R a set of *defining relators* of G . Notice that if $r \in R$

$$r\phi = 1.$$

We often express this fact by writing

$$r = 1 \quad \text{in } G .$$

More generally if w is any reduced X -product such that

$$w\varphi = 1$$

we term w a *relator*- this in addition to the elements in R , and sometimes say

$$w = 1$$

is a relation in G . This suggests a variety of alternative notations for presentations e.g.

$$G = \langle X ; \{r = 1 \mid r \in R\} \rangle$$

and so on. In all instances such notations only make sense once they are recast in the form discussed above.

Definition 1.1 Suppose $G = \langle X; R \rangle$ and that w is a relator in G . So $w\phi = 1$ i.e., w is in the kernel of ϕ where ϕ denotes the presentation map. We term w a *consequence* of R . Notice

$$w = \prod_i f_i^{-1} r_i^{\pm 1} f_i \quad (f_i \in F, \text{ the free group on } X, r_i \in R) .$$

So every relation $w = 1$ in G is a "consequence" of the defining relations of G .

Definition 1.2 We term $G = \langle X; R \rangle$ a *recursive presentation* of G if R is a recursively enumerable subset of $F = F(X)$.

Definition 1.3 A presentation $G = \langle X; R \rangle$ is said to have a *solvable word problem* if $gp_F(R)$ is a recursive subset of $F = F(X)$. This means that both $gp_F(R)$ and its complement in F are recursively enumerable. A group G is said to have a *solvable word problem* if it has a presentation with a solvable word problem.

Of course a given group can well have lots of different presentations e.g. here is a presentation of a free group:

$$G = \langle x_1, x_2, \dots ; \{x_{f(i)} = 1 \mid i = 1, 2, \dots\} \rangle \quad (*)$$

where f is a function from the positive integers into the positive integers. If f is recursive then this presentation is a recursive presentation; and if the complement of the range of f is not a recursively enumerable subset of the positive integers, then $*$ is a recursive presentation of a free group (incidentally necessarily of infinite rank) with an insoluble word problem. Of course a free group of countably infinite rank has a recursive presentation with a solvable word problem. So given two recursive presentations of a group, deducing its structure algorithmically depends heavily on the presentations involved.

We shall show that if we restrict our attention to finite presentations this behaviour cannot occur i.e. if a group G is given by two finite presentations then either both presentations have a solvable word problem or neither does. This means that having a solvable word problem is an invariant of finite presentations, i.e. having a solvable word problem depends on the group itself and we express this by saying that having a solvable word problem is an algebraic property of finite presentations of a given group.

Lemma 1.4 *Let $G = \langle X; R \rangle$ be a recursive presentation of G . Then the consequences of R form a recursively enumerable set.*

Proof. By assumption

$$X = \{ x_1, x_2, \dots \}$$

and R is a recursively enumerable subset of the free group F on X . So we can list R and hence all products of conjugates of R and their inverses i.e. $gp_F(R)$ is a recursively enumerable subset of the free group on X .

This preoccupation with presentations of this kind is no idle pursuit – as I have already noted they play a critical role in the subgroup structure of finitely presented groups.

1.1 Tietze transformations

There are four simple ways of going from one presentation of a group G to another, called Tietze transformations, which we formulate in the form of a proposition.

Proposition 1.5 *Let G be a group. Then the following hold:*

1. T1. *If $G = \langle X; R \rangle$ then $G = \langle X \dot{\cup} Y; R \cup \{ yw(x)^{-1} \mid y \in Y \} \rangle$ where here $w(x)$ is a reduced X -product corresponding to $y \in Y$ i.e. different y 's may well have different $w(x)$'s attached to them.*

2. $T1'$. If $G = \langle X \dot{\cup} Y; R \cup \{yw(x)^{-1} \mid y \in Y\} \rangle$ then $G = \langle X; R \rangle$.
3. $T2$. If $G = \langle X; R \rangle$ and S is a set of consequences of R , then $G = \langle X; R \cup S \rangle$.
4. $T2'$. If $G = \langle X; R \cup S \rangle$ and S is a set of consequences of R , then $G = \langle X; R \rangle$.

$T1$ and $T1'$ are called *Tietze transformations of type 1*; $T2$ and $T2'$ are called *Tietze transformations of type 2*. Once one interprets these transformations, the proofs are easy. Consider, for example, $T1$. We have

$$G = \langle X; R \rangle .$$

So X comes with its presentation map ϕ . We infer, although this information is not given explicitly, that the presentation map ϕ^+ involved in the presentation

$$\langle X \cup Y; R \cup \{yw(x)^{-1} \mid y \in Y\} \rangle$$

is ϕ on X and that

$$y\phi^+ = w(x\phi).$$

Our objective is to prove that ϕ^+ is onto and that the kernel of ϕ^+ is

$$\ker \phi^+ = gp_{F^+}(R \cup \{yw(x)^{-1} \mid y \in Y\}) , \quad (**)$$

where

$$F = F(X) \text{ and } F^+ = F(X \cup Y).$$

It is obvious that ϕ^+ is onto. To prove $(**)$ we observe first that

$$F^+ \text{ is free on } X \cup \tilde{Y} ,$$

where

$$\tilde{Y} = \{yw(x)^{-1} \mid y \in Y\} .$$

We define then ϕ^+ to be the map which sends the elements $yw(x)^{-1}$ to the identity and the elements of $x \in X$ to $x\phi$. It follows that

$$\ker \phi^+ = gp_{F^+}(R \cup \{yw(x)^{-1} \mid y \in Y\})$$

as required.

The use of these Tietze transformations will be demonstrated below in the proof of the following

Theorem 1.6 (Tietze) *Let*

$$G = \langle X; R \rangle \text{ and also } G = \langle Y; S \rangle .$$

Then the first presentation can be transformed into the second by Tietze transformations.

This shows that finite presentations of a given group are closely connected, even though this can not be detected algorithmically. Finally we have a particularly useful observation due to von Dyck.

1.2 von Dyck's lemma

Lemma 1.7 (von Dyck's lemma.) *Suppose that $G = \langle X; R \rangle$ with presentation map $\phi : X \rightarrow G$. Furthermore, let H be a second group and suppose that $\theta : X \rightarrow H$. If $r(x_1\phi\theta, \dots, x_n\phi\theta) = 1$ whenever $r(x_1, \dots, x_n) \in R$, then θ can be continued to a homomorphism of G into H .*

The proof of von Dyck's lemma is not hard. Noticed first that the map ϕ gives rise to a homomorphism of $F = F(X)$ onto G with kernel $gp_F(R)$. If we now observe that the continuation of the map $\phi\theta$ from X into H to F maps R to the identity, then this induces a homomorphism from $G \cong F/gp_F(R)$ into H , as claimed.

1.3 Some exercises and proofs

1. Let

$$G = \langle x_1, x_2, \dots ; \{x_{f(i)} = 1 \mid i = 1, 2, \dots\} \rangle \quad (*)$$

where f is a function from the positive integers \mathbb{P} into the positive integers. We claimed that G is a free group. To see this, let

$$Y = \{x_i \mid i \in \{f(i) \mid i \in \mathbb{P}\}\}$$

and let

$$Z = \mathbb{P} - Y.$$

Then the presentation for G can be written in the form:

$$G = \langle Y \cup Z; x_j = 1 (z_j \in Z) \rangle .$$

It follows then from a Tietze transformation of Type II', that

$$G = \langle Y; \rangle ,$$

i.e., G is defined on a set of generators with an empty set of defining relators, which means that it is free.

2. Let's use these Tietze transformations to prove that if

$$G = \langle X; R \rangle \text{ and also } G = \langle Y; S \rangle .$$

then the first presentation can be transformed into the second by Tietze transformations.

For ease of notation we identify the elements of X and Y as elements of G . Since X generates G each of the elements $y \in Y$ can be expressed in terms of the generators X :

$$\{y^{-1}w_y(\underline{x}) \mid y \in Y\}.$$

So by T1

$$G = \langle X \cup Y; R \cup S \cup \{y^{-1}w_y(\underline{x}) \mid y \in Y\} \rangle .$$

But Y also generates G . So the elements $x \in X$ can also be written in terms of the elements of Y :

$$\{x^{-1}v_x(\underline{y}) \mid x \in X\}.$$

So, by T1 we have

$$G = \langle X \cup Y; R \cup S \cup \{y^{-1}w_y(\underline{x}) \mid y \in Y\} \cup \{x^{-1}v_x(\underline{y}) \mid x \in X\} \rangle . \quad (1)$$

Now each of the elements of R are words in the elements of X and since each of the elements of X have been expressed as words in the elements of Y we can add to the presentation (1) the entire set of relators R with each of the elements x replaced by the $v_x(\underline{y})$ which gives rise to a set \underline{R} and the presentation

$$G = \langle X \cup Y; R \cup S \cup \{y^{-1}w_y(\underline{x}) \mid y \in Y\} \cup \{x^{-1}v_x(\underline{y}) \mid x \in X\} \cup \underline{R} \rangle . \quad (1)$$

Now we can use a Tietze transformation of Type I to remove the set of relators R and then a Tietze transformation of Type I to remove both \underline{R} and X leaving us with $\langle Y; S \rangle$ as desired.

1.4 Examples

In general it is difficult to find presentations of given groups and to verify that these are indeed presentations of the groups involved.

Find presentations of the following groups:

1. Find a presentation of the symmetric groups of degree 3 and 4.
2. Find a presentation of the symmetric group of degree n .